Physics 604

Final Exam

Dec. 16, 2010

1) (30 pts) Four point charges with charge $\pm q$ are arranged as in Figure 1.

a) (5 pts.) What is the charge density function $\rho(r,\theta,\phi)$?

$$\rho(r,\theta,\phi) = -q\delta(r-a)\delta(\cos\theta)\left[\delta(\phi)-\delta(\phi-\pi/2)+\delta(\phi-\pi)-\delta(\phi-3\pi/2)\right]/r^2$$

b) (5 pts.) What are the multipoles q_{lm} (do as many as you can; there is useful information in the HW solutions)? In particular, what is the first non-vanishing multipole?

$$\begin{aligned}
q_{lm} &= \int r'^l Y_{lm}^*(\theta',\phi')\rho(r',\theta',\phi')r'^2 dr' d\Omega' \\
&= -qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-e^{-im\pi/2}+e^{-im\pi}-e^{-3i\pi/2}\right] \\
&= -qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-e^{-im\pi/2}\right]\left[1+(-1)^m\right] \\
&= -qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-(-i)^m\right]\left[1+(-1)^m\right] \\
&= -qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)2\left[1-(-1)^{m/2}\right] \qquad \text{for } m \text{ even}
\end{aligned}$$

$P_l^m(0)$ vanishes unless $l+m$ is even so l must be even also. The first non-vanishing multipole needs to have $m=2$ and is

$$q_{22} = -qa^2\sqrt{\frac{5}{4\pi}\frac{1}{4!}}4\cdot 3 = -qa^2\sqrt{\frac{15}{2\pi}}$$

c) (4 pts.) Do multipoles for odd m always vanish? Can you explain why or why not using the rotational symmetry in the source?

Yes. Notice that a rotation about the z-axis by π leaves the charge density identical to the initial density, i.e. $\rho(r,\theta,\phi+\pi)=\rho(r,\theta,\phi)$. But under rotation by π

$$Y_{lm}^*(\theta,\phi+\pi) = Y_{lm}^*(\theta,\phi)e^{-im\pi} = (-1)^m Y_{lm}^*(\theta,\phi)$$

Therefore, for rotationally symmetrical distributions

$$
\begin{aligned}
q_{lm} &= \int r'^l Y_{lm}^*(\theta',\phi')\rho(r',\theta',\phi')r'^2 dr' d\Omega' \\
&= \int r'^l Y_{lm}^*(\theta',\phi'+\pi)\rho(r',\theta',\phi'+\pi)r'^2 dr' d\Omega' \\
&= (-1)^m \int r'^l Y_{lm}^*(\theta',\phi')\rho(r',\theta',\phi')r'^2 dr' d\Omega' \\
&= (-1)^m q_{lm}
\end{aligned}
$$

For m odd, the multipole moments must vanish.

d) (5 pts.) What is the potential at locations with $r > a$ and $r < a$?

By the addition theorem

$$
\Phi(r,\theta,\phi) = \frac{q}{4\pi\varepsilon_0} \sum_{\substack{l=0 \\ even}}^{\infty} \sum_{\substack{m=-l \\ even}}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} \left[-Y_{lm}^*(0,0) + Y_{lm}^*(0,\pi/2) - Y_{lm}^*(0,\pi) + Y_{lm}^*(0,3\pi/2) \right] Y_{lm}(\theta,\phi)
$$

$$
= -\frac{2q}{\varepsilon_0} \sum_{\substack{l=0 \\ even}}^{\infty} \sum_{\substack{m=-l \\ even}}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - (-1)^{m/2} \right] Y_{lm}(\theta,\phi)
$$

where $r_<$ and $r_>$ are the smaller and larger of r and a.

e) (5 pts.) Suppose the charges are enclosed in a grounded conducting sphere of radius b centered on the origin. What are the potentials for $r > a$ and $r < a$?

The easiest method is to simply add to the solution in d) the negative of the solution of Laplace's equation that has the correct boundary potential value

$$
\begin{aligned}
\Phi_{tot}(r,\theta,\phi) = &-\frac{2q}{\varepsilon_0} \sum_{\substack{l=0 \\ even}}^{\infty} \sum_{\substack{m=-l \\ even}}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - (-1)^{m/2} \right] Y_{lm}(\theta,\phi) \\
&+ \frac{2q}{\varepsilon_0} \sum_{\substack{l=0 \\ even}}^{\infty} \sum_{\substack{m=-l \\ even}}^{l} \frac{1}{2l+1} \frac{a^l r^l}{b^{2l+1}} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - (-1)^{m/2} \right] Y_{lm}(\theta,\phi)
\end{aligned}
$$

$$
= -\frac{2q}{\varepsilon_0} \sum_{\substack{l=0 \\ even}}^{\infty} \sum_{\substack{m=-l \\ even}}^{l} \frac{1}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{a^l r^l}{b^{2l+1}} \right] \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(0) \left[1 - (-1)^{m/2} \right] Y_{lm}(\theta,\phi)
$$

where $r_<$ and $r_>$ are the smaller and larger of r and a. This solution would also be obtained by adding the potentials of the four image charges with strength $q' = \mp qb/a$ at locations $r' = b^2/a$.

f) (4 pts.) For the solution in e), what is the charge density on the inside surface of the sphere?

$\sigma = \varepsilon_0 E_r$

$$E_r = -\left.\frac{\partial \Phi}{\partial r}\right|_{r=b} = \frac{2q}{\varepsilon_0}\sum_{\substack{l=0 \\ even}}^{\infty}\sum_{\substack{m=-l \\ even}}^{l}\frac{1}{2l+1}\left[-(l+1)\frac{a^l}{b^{l+2}} - l\frac{a^l b^{l-1}}{b^{2l+1}}\right]\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-(-1)^{m/2}\right]Y_{lm}(\theta,\phi)$$

$$\sigma = -\frac{2q}{b^2}\sum_{\substack{l=0 \\ even}}^{\infty}\sum_{\substack{m=-l \\ even}}^{l}\frac{a^l}{b^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1-(-1)^{m/2}\right]Y_{lm}(\theta,\phi)$$

g) (2 pts.) What is the field outside the sphere?

Because the charge is inside the conducting sphere, by Gauss's Law the field outside the sphere is shielded away to zero.

2) (25 pts.) Solve the square 2-D potential problem given by Figure 2. The following steps may be helpful.

a) (5 pts.) What are the fundamental solutions in Cartesian coordinates?

An example of an expansion set is

$$\Phi(x,y) = \sum_{m=0}^{\infty} \begin{matrix} A_m \sin(m\pi x/a)e^{m\pi y/a} + B_m \cos(m\pi x/a)e^{m\pi y/a} + C_m \sin(m\pi x/a)e^{-m\pi y/a} + D_m \cos(m\pi x/a)e^{-m\pi y/a} \\ E_m \sin(m\pi y/a)e^{m\pi x/a} + F_m \cos(m\pi y/a)e^{m\pi x/a} + G_m \sin(m\pi y/a)e^{-m\pi x/a} + H_m \cos(m\pi y/a)e^{-m\pi x/a} \end{matrix}$$

b) (5 pts) Solve the boundary value problem with the potential Φ at $x=0$, $0<y<a$ and $x=a$, $0<y<a$ equal to V and with the other sides having $\Phi = 0$.

Here because of the boundary condition at $y=0$ and $y=a$

$$\Phi(x,y) = \sum_{m=0}^{\infty} E_m \sin(m\pi y/a)e^{m\pi x/a} + G_m \sin(m\pi y/a)e^{-m\pi x/a}$$

Because of the boundary conditions at $x=0$ and $x=a$

$$V = \sum_{m=1}^{\infty} E_m \sin(m\pi y/a) + G_m \sin(m\pi y/a)$$

$$V = \sum_{m=1}^{\infty} E_m \sin(m\pi y/a)e^{m\pi} + G_m \sin(m\pi y/a)e^{-m\pi}$$

Orthogonality gives

$$\frac{Va}{m\pi}\left(-\cos m\pi+1\right)=\frac{a}{2}\left(E_m+G_m\right)$$

$$\frac{Va}{m\pi}\left(-\cos m\pi+1\right)=\frac{a}{2}\left(E_m e^{m\pi}+G_m e^{-m\pi}\right)$$

$$\therefore E_m=\frac{2V}{m\pi}\left(-\cos m\pi+1\right)\left(1-e^{m\pi}\right)/\left(1-e^{2m\pi}\right)=-\frac{V}{m\pi}\left(-\cos m\pi+1\right)\left(e^{-m\pi}-1\right)/\sinh m\pi$$

$$G_m=\frac{2V}{m\pi}\left(-\cos m\pi+1\right)\left(1-e^{-m\pi}\right)/\left(1-e^{-2m\pi}\right)=\frac{V}{m\pi}\left(-\cos m\pi+1\right)\left(e^{m\pi}-1\right)/\sinh m\pi$$

$$\Phi_1\left(x,y\right)=\sum_{\substack{m=1\\odd}}^{\infty}\frac{4V}{m\pi}\frac{1}{\sinh m\pi}\sin\left(m\pi y/a\right)\left(\sinh m\pi x/a-\sinh m\pi\left[\left(x-a\right)/a\right]\right)$$

Note that m must be odd for a non-zero expansion coefficient.

c) (5 pts.) Solve an appropriate boundary value problem to include the faces at $\Phi=-V$.

The solution is the same switching $x\leftrightarrow y$ and reversing the sign of the face values:

$$\Phi_2\left(x,y\right)=-\sum_{\substack{m=1\\odd}}^{\infty}\frac{4V}{m\pi}\frac{1}{\sinh m\pi}\sin\left(m\pi x/a\right)\left(\sinh m\pi y/a-\sinh m\pi\left[\left(y-a\right)/a\right]\right)$$

d) (5 pts.) What is the total potential?

By superposition

$$\Phi_{tot}\left(x,y\right)=\Phi_1\left(x,y\right)+\Phi_2\left(x,y\right)$$
$$=\sum_{\substack{m=1\\odd}}^{\infty}\frac{4V}{m\pi}\frac{1}{\sinh m\pi}\begin{bmatrix}\sin\left(m\pi y/a\right)\left(\sinh m\pi x/a-\sinh m\pi\left[\left(x-a\right)/a\right]\right)\\-\sin\left(m\pi x/a\right)\left(\sinh m\pi y/a-\sinh m\pi\left[\left(y-a\right)/a\right]\right)\end{bmatrix}$$

e) (5 pts.) Put a primed coordinate system with its origin at $\vec{x}=\left(a/2,a/2\right)$. Near $r'=0$ what is the θ' dependence of the potential? Near $r'=0$ what is the θ' dependence of the electric field?

3) (25 pts.) A uniformly charged infinitely thin spherical shell of charge q and radius a is spun around the z - axis with constant angular frequency ω.

a) (5 pts.) Show that the current density function $\vec{J}\left(r,\theta,\phi\right)$ is

$$\vec{J}\left(r,\theta,\phi\right)=\frac{q\omega}{4\pi a}\delta\left(r-a\right)\sin\theta\hat{\phi}.$$

The easy way to get this is to realize the charge density is

$$\rho = \frac{q}{4\pi a^2}\delta(r-a),$$

and the rotation velocity is $\vec{v} = \omega a \sin\theta\hat{\phi}$. Multiplying these two results gives the expression. Another way is to consider the current passing through the area element $rdrd\theta$ at $r = a$ and polar angle θ. The total charge in the ring of charge at this location is $(q/2)\sin\theta d\theta$. This charge passes the location on the sphere at θ with frequency $\omega/2\pi$, so the local current is $(q\omega/4\pi)\sin\theta d\theta$. The normal to the area element is $\hat{\phi}$. Now

$$\int J_\phi r dr d\theta = \frac{q\omega}{4\pi}\sin\theta d\theta \to J_\phi = \frac{q\omega}{4\pi a}\delta(r-a)\sin\theta$$

b) (5 pts.) What is the vector potential in all space for this current source assuming the vector potential vanishes as $r \to \infty$?

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' = \frac{\mu_0 q\omega}{16\pi^2 a}\int\frac{\delta(r'-a)\sin\theta'}{|\vec{x}-\vec{x}'|}\hat{\phi}' d^3x'$$

$$A_\phi = \vec{A}\cdot\hat{\phi} = \frac{\mu_0 q\omega}{16\pi^2 a}\int\frac{\delta(r'-a)\sin\theta'}{|\vec{x}-\vec{x}'|}\left[\cos\phi\cos\phi' + \sin\phi\sin\phi'\right]r'^2 dr' d\Omega'$$

$$= -\sqrt{\frac{8\pi}{3}}\frac{\mu_0 q\omega}{32\pi^2 a}\int\frac{\delta(r'-a)}{|\vec{x}-\vec{x}'|}\begin{bmatrix}\cos\phi\left(Y_{11}(\theta',\phi') - Y_{1-1}(\theta',\phi')\right)\\ +\sin\phi\left(Y_{11}(\theta',\phi') + Y_{1-1}(\theta',\phi')\right)/i\end{bmatrix}r'^2 dr' d\Omega'$$

$$= -\sqrt{\frac{8\pi}{3}}\frac{\mu_0 q\omega}{3\cdot 8\pi a}\int\delta(r'-a)\frac{r_<^1}{r_>^2}\begin{bmatrix}\cos\phi\left(Y_{11}(\theta,\phi) - Y_{1-1}(\theta,\phi)\right)\\ +\sin\phi\left(Y_{11}(\theta,\phi) + Y_{1-1}(\theta,\phi)\right)/i\end{bmatrix}r'^2 dr'$$

$$= \frac{\mu_0 q\omega\sin\theta}{3\cdot 4\pi a}\int\delta(r'-a)\frac{r_<^1}{r_>^2}r'^2 dr'$$

$$A_\phi = \begin{cases}\dfrac{\mu_0 q\omega r\sin\theta}{3\cdot 4\pi a} & r < a\\[2ex] \dfrac{\mu_0 q\omega a^2\sin\theta}{3\cdot 4\pi r^2} & r > a\end{cases}$$

c) (5 pts.) What is the magnetic induction inside the shell?

$$
\begin{aligned}
\vec{B} = \vec{\nabla}\times\vec{A} &= \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) \\
&= \frac{2\mu_0 q\omega}{3\cdot 4\pi a}\begin{bmatrix} \cos\theta\left(\sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}\right) \\ -\sin\theta\left(\cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}\right) \end{bmatrix} \\
&= \frac{2\mu_0 q\omega}{3\cdot 4\pi a}\hat{z} = \mu_0\frac{2\sigma\omega a}{3}\hat{z}
\end{aligned}
$$

where σ is the (uniform!) surface charge density.

d) (5 pts.) What is the magnetic induction outside the shell and the magnetic moment?

$$
\vec{B} = \frac{\mu_0 q\omega a^2}{3\cdot 4\pi r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] = \mu_0\frac{\sigma\omega a^4}{3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right],
$$

a classic dipole field distribution. Comparing to Eqn. 5.41, the moment is $q\omega a^2/3$. This result could also be obtained by direct integration too.

$$
\begin{aligned}
dm &= dI\pi a^2\sin^2\theta \\
dI &= \frac{\omega}{2\pi}dq = \frac{\omega}{2\pi}\frac{q}{4\pi a^2}a^2 d\cos\theta d\phi \\
\therefore m &= \frac{\omega q a^2}{4}\int_{-1}^{1}\sin^2\theta d\cos\theta = \frac{\omega q a^2}{4}\left[2 - 2/3\right] = \frac{\omega q a^2}{3}
\end{aligned}
$$

e) (5 pts.) What is the magnetic energy inside and outside the shell?

The magnetic energy inside the shell is easy

$$
T = \frac{\vec{B}\cdot\vec{B}}{2\mu_0}V = \frac{\mu_0^2}{2\mu_0}\left(\frac{2\sigma\omega a}{3}\right)^2\left(\frac{4\pi a^3}{3}\right) = \frac{8\pi\mu_0\sigma^2\omega^2 a^5}{27}.
$$

The magnetic energy outside the shell is only slightly more taxing

$$
\begin{aligned}
T &= \int \frac{\vec{B}\cdot\vec{B}}{2\mu_0} d^3x \\
&= \frac{\mu_0^2}{2\mu_0}\left(\frac{\sigma\omega a^4}{3}\right)^2 2\pi \int_a^\infty \int_{-1}^{1} \frac{4\cos^2\theta + \sin^2\theta}{r^4} dr d\cos\theta \\
&= \frac{\mu_0}{2}\left(\frac{\sigma\omega a^4}{3}\right)^2 2\pi \int_a^\infty \int_{-1}^{1} \frac{3\cos^2\theta + 1}{r^4} dr d\cos\theta \\
&= \frac{\mu_0}{2}\left(\frac{\sigma\omega a^4}{3}\right)^2 8\pi \int_a^\infty \frac{dr}{r^4} \\
&= \frac{\mu_0}{2}\left(\frac{\sigma\omega a^4}{3}\right)^2 \frac{8\pi}{3a^3},
\end{aligned}
$$

one-half the energy inside!

4) (20 pts.) In our final homework set we dealt with accelerator dipole magnets. In this problem, solve a similar problem for 2-D accelerator quadrupole magnets with a model current density

$$
J_z(r,\theta) = \frac{NI}{a}\cos 2\theta\,\delta(r-a).
$$

a) (4 pts.) What is the vector potential generated by this source assuming the vector potential vanishes as $r \to \infty$?

The vector potential is

$$
A_z(r,\theta) = \begin{cases} \sum_{m=0}^{\infty}\left(A_m r^m \sin m\theta + B_m r^m \cos m\theta\right) & r < a \\ \sum_{m=0}^{\infty}\left(C_m r^{-m} \sin m\theta + D_m r^{-m} \cos m\theta\right) & r > a \end{cases}
$$

Continuity and orthogonality at $r = a$ imply

$$
A_z(r,\theta) = \begin{cases} \sum_{m=0}^{\infty}\left(A_m r^m \sin m\theta + B_m r^m \cos m\theta\right) & r < a \\ \sum_{m=0}^{\infty}\left(A_m a^{2m} r^{-m} \sin m\theta + B_m a^{2m} r^{-m} \cos m\theta\right) & r > a \end{cases}
$$

The jump condition at $r = a$ and orthogonality imply $A_m = 0$, $B_m = 0$ for $m \neq 2$, and

$$r\left.\frac{\partial A_z}{\partial r}\right|_{r+\varepsilon} - r\left.\frac{\partial A_z}{\partial r}\right|_{r-\varepsilon} = -\mu_0 NI\cos 2\theta$$

$$B_2 a^4(-2)a^{-2} - B_2 2a^2 = -(\mu_0 NI)$$

$$B_2 = \frac{\mu_0 NI}{4a^2}$$

Therefore

$$A_z(r,\theta) = \begin{array}{ll} \frac{\mu_0 NI}{4}\frac{r^2}{a^2}\cos 2\theta & r<a \\ \frac{\mu_0 NI}{4}\frac{a^2}{r^2}\cos 2\theta & r>a \end{array}$$

b) (4 pts.) What is the magnetic induction inside and outside $r = a$?

Taking the curl of the vector potential

$$B_r(r,\theta) = \frac{1}{r}\frac{\partial A_z}{\partial \theta} = \begin{array}{ll} -\frac{\mu_0 NI}{2}\frac{r}{a^2}\sin 2\theta & r<a \\ -\frac{\mu_0 NI}{2}\frac{a^2}{r^3}\sin 2\theta & r>a \end{array}$$

$$B_\theta(r,\theta) = -\frac{\partial A_z}{\partial r} = \begin{array}{ll} -\frac{\mu_0 NI}{2}\frac{r}{a^2}\cos 2\theta & r<a \\ \frac{\mu_0 NI}{2}\frac{a^2}{r^3}\cos 2\theta & r>a \end{array}$$

c) (4 pts.) Write the induction for $r < a$ in terms of the Cartesian coordinates x and y and the Cartesian unit vectors $\hat{x}$ and $\hat{y}$. What are the powers of x and/or y that appear?

$$\vec{B} = B_r\hat{r} + B_\theta\hat{\theta}$$

$$= -\frac{\mu_0 NI}{2a^2}\left[r2\sin\theta\cos\theta(\cos\theta\hat{x} + \sin\theta\hat{y}) + r(\cos^2\theta - \sin^2\theta)(-\sin\theta\hat{x} + \cos\theta\hat{y})\right]$$

$$= -\frac{\mu_0 NI}{2a^2}\left[r\sin\theta\hat{x} + r\cos\theta\hat{y}\right] = -\frac{\mu_0 NI}{2a^2}\left[y\hat{x} + x\hat{y}\right]$$

The field is *linear* in the Cartesian variables.

d) (4 pts.) What is the magnetic energy per unit length inside and outside $r = a$?

The magnetic energy inside $r = a$

$$T' = \int \frac{\vec{B}\cdot\vec{B}}{2\mu_0} r dr d\theta$$
$$= \frac{\mu_0}{2}\left(\frac{NI}{2a^2}\right)^2 2\pi \int_0^a r^3 dr$$
$$= \frac{\mu_0}{2}\left(\frac{NI}{2a^2}\right)^2 2\pi \frac{a^4}{4} = \pi\mu_0 \frac{N^2 I^2}{16}$$

The magnetic energy outside $r = a$

$$T' = \frac{\mu_0}{2}\left(\frac{NIa^2}{2}\right)^2 \int_a^\infty \int_0^{2\pi} \left(\sin^2 2\theta + \cos^2 2\theta\right) r^{-5} dr d\theta$$
$$= \frac{\mu_0}{2}\left(\frac{NIa^2}{2}\right)^2 2\pi \frac{1}{4a^4} = \pi\mu_0 \frac{N^2 I^2}{16},$$

the same as inside (this is a result like Problem 5.30 in Jackson!)

e) (4 pts.) What is the self inductance per unit length of the magnet assuming I is the total current entering (and leaving!) the magnet?

$$\frac{1}{2} L' I^2 = T' = \pi\mu_0 \frac{N^2 I^2}{16} + \pi\mu_0 \frac{N^2 I^2}{16} = \pi\mu_0 \frac{N^2 I^2}{8}$$
$$\therefore L' = \pi\mu_0 \frac{N^2}{8}$$

Extra credit (5 pts.): Given what you already know, what is a functional form of the potential as a function of r and θ, and also of x and y for a sextupole magnet? (you've done di(two)poles and quad(four)rupoles already now!)

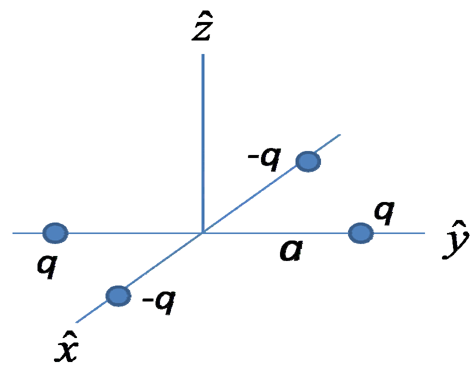

Figure 1

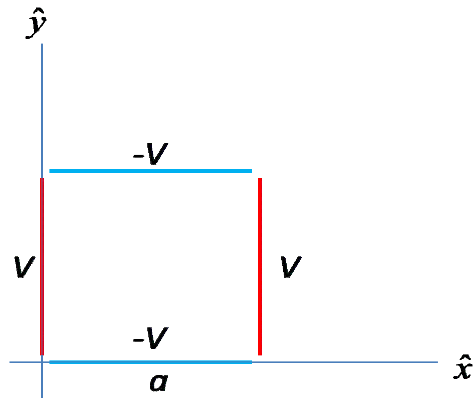

Figure 2